Math 150, *Lecture Notes- Bonds* Name____________

Section 5.1 The Natural Logarithmic Function: Differentiation

The Natural Logarithmic Function

Recall that the General Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$ General Power Rule

has an important disclaimer—it doesn't apply when $n = -1$. Consequently, you have not yet found an antiderivative for the function $f(x) = 1/x$. In this section, you will use the Second Fundamental Theorem of Calculus to *define* such a function. This antiderivative is a function that you have not encountered previously in the text. It is neither algebraic nor trigonometric, but falls into a new class of functions called *logarithmic functions*. This particular function is the **natural logarithmic function.**

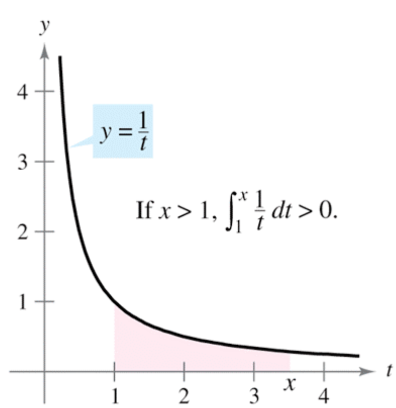
Definition of the Natural Logarithmic Function

The **natural logarithmic function** is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0.$$

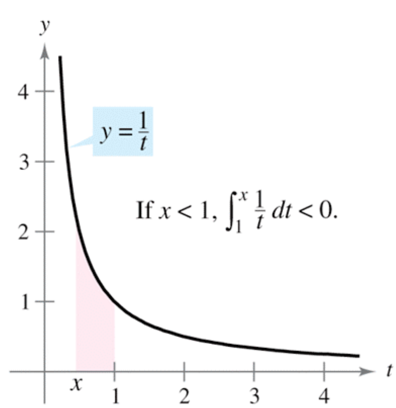
The domain of the natural logarithmic function is the set of all positive real numbers.

From this definition, you can see that $\ln x$ is positive for $x > 1$ and negative for $0 < x < 1$, as shown in Figure 5.1. Moreover, $\ln(1) = 0$, because the upper and lower limits of integration are equal when $x = 1$.



If $x > 1$, then $\ln x > 0$.

Figure 5.1

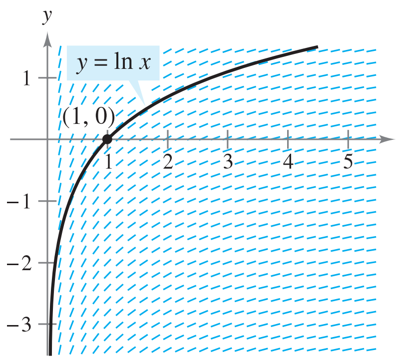


If $0 < x < 1$, then $\ln x < 0$.

To sketch the graph of $y = \ln x$, you can think of the natural logarithmic function as an *antiderivative* given by the differential equation

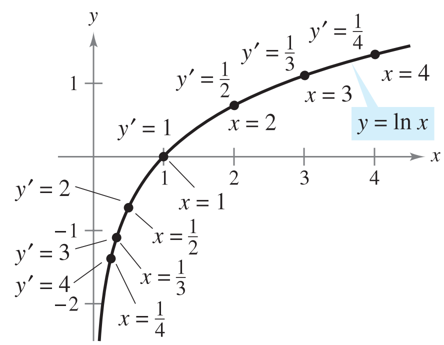
$$\frac{dy}{dx} = \frac{1}{x}.$$

Figure 5.2 is a computer-generated graph, called a *slope (or direction) field*, showing small line segments of slope $1/x$. The graph of $y = \ln x$ is the solution that passes through the point $(1, 0)$.



Each small line segment has a slope of $\frac{1}{x}$.

Figure 5.2



The natural logarithmic function is increasing, and its graph is concave downward.

Figure 5.3

$$\lim_{x\to 0^+} \ln x = -\infty \quad \text{and} \quad \lim_{x\to\infty} \ln x = \infty$$

THEOREM 5.1 Properties of the Natural Logarithmic Function

The natural logarithmic function has the following properties.

1. The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.
2. The function is continuous, increasing, and one-to-one.
3. The graph is concave downward.

Using the definition of the natural logarithmic function, you can prove several important properties involving operations with natural logarithms. If you are already familiar with logarithms, you will recognize that these properties are characteristic of all logarithms.

THEOREM 5.2 Logarithmic Properties

If a and b are positive numbers and n is rational, then the following properties are true.

1. $\ln(1) = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

Ex.1 Expanding Logarithmic Expressions

a. $\ln\frac{10}{9} =$ $\ln(10) - \ln(9)$

$= \ln(10) - \ln(3^2)$

$= \ln(10) - 2\cdot\ln(3)$

b. $\ln\sqrt{3x+2} =$ $\ln(3x+2)^{1/2}$

$= \frac{1}{2}\cdot\ln(3x+2)$

c. $\ln\frac{6x}{5} =$ $\ln(\underline{6x}) - \ln(5)$

$= \ln(6) + \ln(x) - \ln(5)$

d. $\ln\frac{(x^2+3)^2}{x\sqrt[3]{x^2+1}} =$ $\ln(x^2+3)^2 - \ln x\cdot\sqrt[3]{x^2+1}$

$= 2\cdot\ln(x^2+3) - [\ln(x) + \ln\sqrt[3]{x^2+1}]$

$= 2\ln(x^2+3) - \ln(x) - \ln(x^2+1)^{1/3}$

$= 2\ln(x^2+3) - \ln(x) - \frac{1}{3}\ln(x^2+1)$

The Number e

It is likely that you have studied logarithms in an algebra course. There, without the benefit of calculus, logarithms would have been defined in terms of a **base** number. For example, common logarithms have a base of 10 and therefore $\log_{10} 10 = 1$. (You will learn more about this in Section 5.5.)

The **base for the natural logarithm** is defined using the fact that the natural logarithmic function is continuous, is one-to-one, and has a range of $(-\infty, \infty)$. So, there must be a unique real number x such that $\ln x = 1$, as shown in Figure 5.5. This number is denoted by the letter e. It can be shown that e is irrational and has the following decimal approximation.

$$e \approx 2.71828182846$$

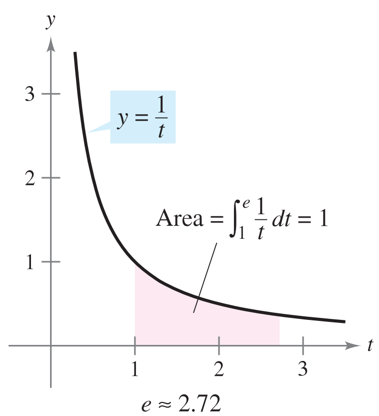
$\ln(e) = \log_e(e^1)$

$= 1$

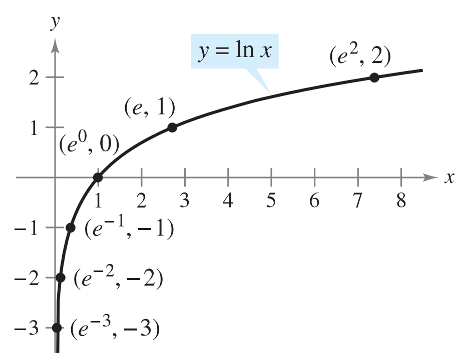
Definition of e

The letter e denotes the positive real number such that

$$\ln e = \int_1^e \frac{1}{t}\,dt = 1.$$



e is the base for the natural logarithm because $\ln e = 1$.
Figure 5.5



If $x = e^n$, then $\ln x = n$.
Figure 5.6

Once you know that $\ln e = 1$, you can use logarithmic properties to evaluate the natural logarithms of several other numbers. For example, by using the property

$$\begin{aligned} \ln(e^n) &= n \ln e \\ &= n(1) \\ &= n \end{aligned}$$

you can evaluate $\ln(e^n)$ for various values of n, as shown in the table and in Figure 5.6.

x	$\frac{1}{e^3} \approx 0.050$	$\frac{1}{e^2} \approx 0.135$	$\frac{1}{e} \approx 0.368$	$e^0 = 1$	$e \approx 2.718$	$e^2 \approx 7.389$
$\ln x$	-3	-2	-1	0	1	2

Ex.2 Evaluating Natural Logarithmic Expressions

a. $\ln 2 \approx$

b. $\ln 32 \approx$

c. $\ln 0.1 \approx$

The Derivative of the Natural Logarithmic Function

The derivative of the natural logarithmic function is given in Theorem 5.3. The first part of the theorem follows from the definition of the natural logarithmic function as an antiderivative. The second part of the theorem is simply the Chain Rule version of the first part.

THEOREM 5.3 Derivative of the Natural Logarithmic Function

Let u be a differentiable function of x.

1. $\dfrac{d}{dx}[\ln x] = \dfrac{1}{x}, \quad x > 0$

2. $\dfrac{d}{dx}[\ln u] = \dfrac{1}{u}\dfrac{du}{dx} = \dfrac{u'}{u}, \quad u > 0$

Ex.3 Differentiation of Logarithmic Functions

a. $\dfrac{d}{dx}[\ln(2x)] = \dfrac{1}{2x} \cdot \underbrace{\dfrac{d}{dx}(2x)}_{\text{chain Rule}}$

$= \dfrac{1}{2x} \cdot 2$

$= \dfrac{1}{x}$

b. $\dfrac{d}{dx}[\ln(x^2 + 1)] = \dfrac{1}{x^2+1} \cdot \underbrace{\dfrac{d}{dx}[x^2+1]}_{\text{chain Rule}}$

$= \dfrac{1}{x^2+1} \cdot 2x$

$= \dfrac{2x}{x^2+1}$

Product Rule

c. $\dfrac{d}{dx}[x \ln x] = x \cdot \dfrac{d}{dx}[\ln(x)] + [\ln(x)] \cdot \dfrac{d}{dx}[x]$

$= x \cdot \dfrac{1}{x} + \ln(x) \cdot 1$

$= 1 + \ln(x)$

chain Rule

d. $\dfrac{d}{dx}[(\ln x)^3] = 3[\ln(x)]^2 \cdot \dfrac{d}{dx}[\ln(x)]$

$= 3[\ln(x)]^2 \cdot \dfrac{1}{x}$

$= \dfrac{3[\ln(x)]^2}{x}$

Napier used logarithmic properties to simplify *calculations* involving products, quotients, and powers. Of course, given the availability of calculators, there is now little need for this particular application of logarithms. However, there is great value in using logarithmic properties to simplify *differentiation* involving products, quotients, and powers.

Ex.4 Logarithmic Properties as Aids to Differentiation

Differentiate $f(x) = \ln\sqrt{x+1}$.

$$f(x) = \ln (x+1)^{1/2}$$
$$f(x) = \frac{1}{2}\ln(x+1)$$
$$f'(x) = \frac{1}{2}\cdot\frac{d}{dx}[\ln(x+1)]$$
$$f'(x) = \frac{1}{2}\cdot\frac{1}{x+1}\cdot\underbrace{\frac{d}{dx}(x+1)}_{\text{chain Rule}}$$
$$f'(x) = \frac{1}{2(x+1)}\cdot 1$$
$$f'(x) = \frac{1}{2(x+1)} \checkmark$$

Ex.5 Logarithmic Properties as Aids to Differentiation

Differentiate $f(x) = \ln\dfrac{x(x^2+1)^2}{\sqrt{2x^3-1}}$.

$$f(x) = \ln[x\cdot(x^2+1)^2] - \ln[\sqrt{2x^3-1}]$$
$$f(x) = \ln(x) + \ln(x^2+1)^2 - \ln(2x^3-1)^{1/2}$$
$$f(x) = \ln(x) + 2\ln(x^2+1) - \frac{1}{2}\ln(2x^3-1) \leftarrow \text{Ready}$$
$$f'(x) = \frac{d}{dx}[\ln(x) + 2\ln(x^2+1) - \frac{1}{2}\ln(2x^3-1)]$$
$$f'(x) = \frac{d}{dx}[\ln(x)] + 2\frac{d}{dx}[\ln(x^2+1)] - \frac{1}{2}\cdot\frac{d}{dx}[\ln(2x^3-1)]$$
$$f'(x) = \frac{1}{x} + 2\cdot\frac{1}{x^2+1}\cdot\underbrace{\frac{d}{dx}[x^2+1]}_{\text{chain Rule}} - \frac{1}{2}\cdot\frac{1}{2x^3-1}\cdot\underbrace{\frac{d}{dx}[2x^3-1]}_{\text{chain Rule}}$$
$$f'(x) = \frac{1}{x} + \frac{2}{x^2+1}\cdot 2x - \frac{1}{2(2x^3-1)}\cdot\frac{2\cdot 3x^2}{1}$$
$$f'(x) = \frac{1}{x} + \frac{4x}{x^2+1} - \frac{3x^2}{2x^3-1}$$

Ex.6 Logarithmic Differentiation

Find the derivative of

$$y = \frac{(x-2)^2}{\sqrt{x^2+1}}, \quad x \neq 2.$$

$$\ln(y) = \ln\left[\frac{(x-2)^2}{\sqrt{x^2+1}}\right]$$

$$\ln(y) = \ln(x-2)^2 - \ln\sqrt{x^2+1}$$

$$\ln(y) = 2\cdot\ln(x-2) - \ln(x^2+1)^{1/2}$$

$$\ln(y) = 2\ln(x-2) - \frac{1}{2}\ln(x^2+1) \leftarrow \text{Ready}$$

$$\frac{d}{dx}\left[\ln(y)\right] = \frac{d}{dx}\left[2\ln(x-2) - \frac{1}{2}\ln(x^2+1)\right]$$

Implicit Differentiation

$$\frac{1}{y}\cdot\frac{dy}{dx} = 2\cdot\frac{d}{dx}\left[\ln(x-2)\right] - \frac{1}{2}\cdot\frac{d}{dx}\left[\ln(x^2+1)\right]$$

chain Rule   chain Rule

$$\frac{1}{y}\cdot\frac{dy}{dx} = 2\cdot\frac{1}{x-2}\cdot\frac{d}{dx}(x-2) - \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot\frac{d}{dx}(x^2+1)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{2}{x-2}\cdot\frac{1}{1} - \frac{1}{2\cdot(x^2+1)}\cdot\frac{2x}{1}$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{2}{x-2} - \frac{x}{x^2+1}$$

$$y\cdot\frac{1}{y}\cdot\frac{dy}{dx} = y\cdot\left[\frac{2}{x-2} - \frac{x}{x^2-1}\right]$$

$$\frac{dy}{dx} = \frac{(x-2)^2}{\sqrt{x^2+1}}\left[\frac{2}{x-2} - \frac{x}{x^2+1}\right]$$

Because the natural logarithm is undefined for negative numbers, you will often encounter expressions of the form $\ln|u|$. The following theorem states that you can differentiate functions of the form $y = \ln|u|$ as if the absolute value notation was not present.

THEOREM 5.4 Derivative Involving Absolute Value

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}[\ln|u|] = \frac{u'}{u}.$$

PROOF If $u > 0$, then $|u| = u$, and the result follows from Theorem 5.3. If $u < 0$, then $|u| = -u$, and you have

$$\begin{aligned}\frac{d}{dx}[\ln|u|] &= \frac{d}{dx}[\ln(-u)] \\ &= \frac{-u'}{-u} \\ &= \frac{u'}{u}.\end{aligned}$$

■

Ex.7 Derivative Involving Absolute Value

Find the derivative of

$$f(x) = \ln|\cos x|.$$

$$f'(x) = \frac{d}{dx}[\ln|\cos(x)|]$$

$$f'(x) = \frac{1}{\cos(x)} \cdot \frac{d}{dx}[\cos(x)]$$

$$f'(x) = \frac{1}{\cos(x)} \cdot [-\sin(x)]$$

$$f'(x) = -\frac{\sin(x)}{\cos(x)}$$

$$f'(x) = -\tan(x)$$

$$\int \tan(x)\,dx$$

$$= -\ln|\cos(x)| + C$$

Ex.8 Finding Relative Extrema

Locate the relative extrema of

$$y = \ln(x^2 + 2x + 3).$$

$\frac{d}{dx}[y] = \frac{d}{dx}[\ln(x^2+2x+3)]$

$\frac{dy}{dx} = \frac{1}{x^2+2x+3} \cdot \frac{d}{dx}[x^2+2x+3]$

Chain Rule

$\frac{dy}{dx} = \frac{1}{x^2+2x+3} \cdot \frac{[2x+2]}{1}$

$$\boxed{\frac{dy}{dx} = \frac{2x+2}{x^2+2x+3}}$$

Find critical numbers:

(A) $\frac{dy}{dx} = 0$

$0 = \frac{2x+2}{x^2+2x+3}$

$0 = 2x+2$

$-2 = 2x$

$-1 = x$ ✓

or

(B) $\frac{dy}{dx}$ is undefined

$x^2+2x+3 = 0$

$a=1$, $b=2$, $c=3$

$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(3)}}{2(1)}$

$x = \frac{-2 \pm \sqrt{4-12}}{2}$

$x = \frac{-2 \pm \sqrt{-8}}{2}$ ← Not a Real Number

None

Use the Second Derivative Test

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{2x+2}{x^2+2x+3}\right]$

$\frac{d^2y}{dx^2} = \frac{(x^2+2x+3)(2) - (2x+2)(2x+2)}{(x^2+2x+3)^2}$

$\frac{d^2y}{dx^2} = \frac{2x^2+4x+6 - (4x^2+8x+4)}{(x^2+2x+3)^2}$

$\frac{d^2y}{dx^2} = \frac{2x^2+4x+6-4x^2-8x-4}{(x^2+2x+3)^2}$

$\frac{d^2y}{dx^2} = \frac{-2x^2-8x+2}{(x^2+2x+3)^2}$

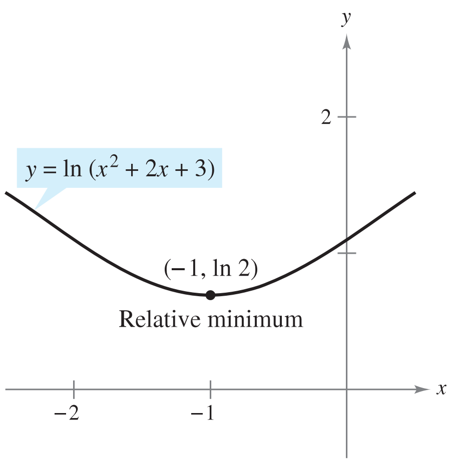
$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = \frac{-2(-1)^2-8(-1)+2}{[(-1)^2+2(-1)+3]^2}$

$= \frac{-2+8+2}{[1-2+3]^2}$

$= \frac{8}{(2)^2}$

$= \frac{8}{4}$

$= 2 > 0$ concave up



The derivative of y changes from negative to positive at $x = -1$.

Figure 5.7